An Averaging Lemma

Let $p(x), q(x)$ be continuous functions defined on an interval $I = [a, b)$, with $p(x) \geq q(x)$. Consider the solutions u, v of the equations

$$u'' + pu = 0 \quad , \quad v'' + qv = 0 \, ,$$

together with the solution w of a "convex combination" of them:

$$w'' + (tp + (1-t)q)\, w = 0 \, ,$$

for some $0 < t < 1$. Assmue u, v, w have the same initial conditions, $u(a) = v(a) = w(a)$, and $u'(a) = v'(a) = w'(a)$.

Lemma 1: *Suppose* $u > 0$ *on* I, *and let* $h = tu + (1-t)v$. *Then*

$$h'' + (tp + (1-t)q)\, h \geq 0 \, ,$$

and hence $h \geq w$.

Proof: By Sturm comparison, we have $v \geq u$ on I. Then

$$h'' + (tp + (1-t)q)\, h = -tpu - (1-t)qv + (tp + (1-t)q)\,(tu + (1-t)v)$$

$$= -tpu - (1-t)qv + t^2 pu + (1-t)^2 qv + t(1-t)(pv + qu)$$

$$= -t(1-t)(pu + qv) + t(1-t)(pv + qu) = t(1-t)(p-q)(v-u) \geq 0 \, .$$

Since h and w have the same initial conditions, it follows again by Sturm comparison that $h \geq w$.

Lemma 1 generalizes to convex combinations of an arbitrary number of equations. Let $p_1 \geq \cdots \geq p_n$ be continuous functions on I, and let $t_1, \ldots, t_n$ be nonnegative numbers with $t_1 + \cdots + t_n = 1$. Consider the solutions u_k, $k = 1, \ldots, n$, of the family of equations

$$u_k'' + p_k u_k = 0 \, ,$$

and w the solution of

$$w + (t_1 p_1 + \cdots + t_n p_n)\, w = 0 \, ,$$

all u_k, w with the same initial conditions at $x = a$. Let $h = t_1 u_1 + \cdots + t_n u_n$.

Lemma 2: *Suppose $u_1 > 0$ on I. Then*

$$h'' + (t_1 p_1 + \ldots + t_n p_n)\, h \geq 0\,,$$

and hence $h \geq w$.

Proof: Because p_1 is the largest coefficient and $u_1 > 0$, it follows that all other functions u_k and w are also positive on I. The proof is by induction. Suppose the lemma is true for combinations of n equations, and let p_k, u_k, t_k, $k = 1, \ldots, n+1$, and w be corresponding quantities to $n+1$ such equations. Let

$$h = t_1 u_1 + \cdots t_n u_n + t_{n+1} u_{n+1} = t_1 u_1 + (1 - t_1) v\,,$$

where

$$v = \frac{1}{1 - t_1}\left(t_2 u_2 + \cdots + t_{n+1} u_{n+1}\right)\,.$$

Note that $v > 0$ on I. By the induction hypothesis

$$v'' + \frac{1}{1 - t_1}\left(t_2 p_2 + \cdots + t_{n+1} p_{n+1}\right) v \geq 0\,,$$

so that

$$v'' + qv = 0$$

for

$$q = -\frac{v''}{v} \;\leq\; \frac{1}{1 - t_1}\left(t_2 p_2 + \ldots + t_{n+1} p_{n+1}\right)\,. \tag{1}$$

In other words,

$$u_1'' + p_1 u_1 = 0 \quad , \quad v'' + qv = 0\,,$$

with $q \leq p_1$. It follows from Lemma 1 that $h = t_1 u_1 + (1 - t_1) v$ satisfies

$$h'' + (t_1 p_1 + (1 - t_1) q)\, h \geq 0\,,$$

and therefore also

$$h'' + (t_1 p_1 + t_2 p_2 + \ldots t_{n+1} p_{n+1})\, h \geq 0\,,$$

because $h > 0$ and (1). The final conclusion $h \geq w$ follows from Sturm comparison.

Finally, let $p_t(x)$ be a decreasing family of continuous functions defined for $x \in I$, varying continuously for $t \in [0, 1]$. Let u_t and w be solutions with the same initial conditions of the equations

$$u_t'' + p_t u_t = 0\,,$$

and

$$w + pw = 0\,,$$

where

$$p(x) = \int_0^1 p_t(x)dx\,.$$

It then follows from Lemma 2 that if $u_0 > 0$ on I, then the function

$$h(x) = \int_0^1 u_t(x)dx$$

satisfies

$$h'' + ph \geq 0\,,$$

and so $h \geq w$.